SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A NONLINEAR DIFFERENTIAL EQUATION BY THE METHOD OF SUCCESSIVE APPROXIMATIONS

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A method is proposed for obtaining an approximate solution of a boundary value problem for the ordinary nonlinear heat-transfer equation in plane Poiseuille flow when the viscosity varies exponentially with the temperature.

It is shown in [1] that the temperature distribution in the flow of a viscous incompressible fluid between parallel plates is described by the following equation in dimensionless quantities:

$$\frac{d^2\Theta}{d\eta^2}+k(\varkappa\eta+C_1)^2\exp\Theta=0 \quad (-1\leqslant\eta\leqslant+1), \tag{1}$$

where the viscosity is assumed to vary exponentially with the temperature. In Eq. (1) k is the coefficient of dissipation of mechanical energy, C_1 is a constant from the first integral of the equation of motion, $\varkappa$ is a coefficient proportional to the longitudinal pressure gradient.

Replacing the independent variable in (1) by the substitution

$$\rho=\frac{\sqrt{k}(\varkappa\eta+C_1)^2}{2\varkappa} \tag{2}$$

reduces it to the form

$$\frac{d^2\Theta}{d\rho^2}+\frac{1}{2\rho}\frac{d\Theta}{d\rho}+\exp\Theta=0. \tag{3}$$

The problem of general Couette flow between parallel plates, two-dimensional Poiseuille flow, the non-isothermal problem of the hydrodynamic theory of lubrication in the Stokes formulation, and other problems reduce to the integration of an equation of type (1). Equation (3) is a special case of a more general equation of the form

$$\frac{d^2Y}{d\rho^2}+\frac{n}{\rho}\frac{dY}{d\rho}+A\exp Y=0, \tag{4}$$

to which a number of astrophysics problems reduce [2, 3], where n, generally speaking, can take on any values. Poincaré [4] showed that this equation is closely related to the theory of Fuchs functions.

The general solution of Eq. (4) for arbitrary n cannot be expressed in terms of known functions [2]. Only for n = 0 and 1 can the solution of this equation be expressed in closed form in terms of known functions [5]. The form of the solutions close to the singular point $\rho = 0$ is investigated in [2] for arbitrary n by using series expansions, but these series are only of theoretical interest since they are too cumbersome to be of practical use.

Apparently the lack of even an approximate analytic solution of the problem of Poiseuille flow in a plane channel is due to the difficulty of integrating Eq. (1).

We show how to obtain an approximate solution of Eq. (1) with a preassigned accuracy by reducing it to a nonlinear integral equation and solving it approximately.

In (1) we make a substitution different from (2) and set

$$\eta=2t-1 \quad (0\leqslant t\leqslant 1), \tag{5}$$

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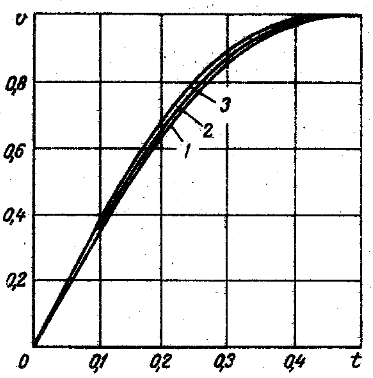

Fig. 1

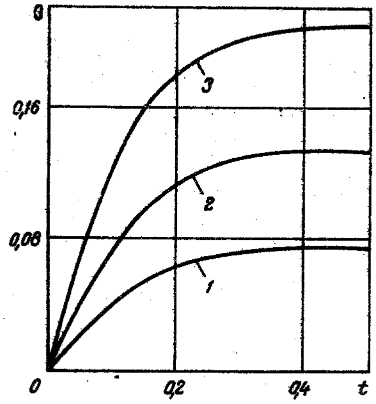

Fig. 2

Fig. 1. Distribution of dimensionless velocity in nonisothermal plane Poiseuille flow. 1) $V_{max} = 3$; 2) 4; 3) 5 m/sec.

Fig. 2. Distribution of dimensionless temperature in nonisothermal plane Poiseuille flow. 1) $V_{max} = 3$; 2) 4; 3) 5 m/sec.

then Eq. (1) and its boundary conditions can be rewritten in the form

$$\frac{d^2\Theta}{dt^2} + (at + b)^2 \exp\Theta = 0, \tag{6}$$

$$\Theta(0) = \Theta_{\mathrm{I}}, \tag{7}$$

$$\Theta(1) = \Theta_{\mathrm{II}}, \tag{8}$$

with the notation $a = 2\sqrt{k}\varkappa$ and $b = \sqrt{k}(C_1 - \varkappa)$.

Integrating Eq. (6) once we obtain

$$\Theta'(t) = -\int_0^t (a\tau + b)^2 \exp\Theta(\tau)\, d\tau + \Theta'(0). \tag{9}$$

Another integration gives

$$\Theta(t) = \Theta(0) + \Theta'(0)\, t - \int_0^t \Big[\int_0^t (a\tau + b)^2 \exp\Theta(\tau)\, d\tau\Big] d\tau, \tag{10}$$

or, satisfying boundary condition (7), we obtain

$$\Theta(t) = \Theta_{\mathrm{I}} + mt - \int_0^t \Big[\int_0^t (a\tau + b)^2 \exp\Theta(\tau)\, d\tau\Big] d\tau, \tag{11}$$

where $\Theta'(0) = m$ is a so-far unknown constant.

We solve the nonlinear integral equation (11) by the method of successive approximations, determining the n-th approximation to the solution of problem (6)-(8) by the expression

$$\Theta_n(t) = \Theta_{\mathrm{I}} + m_n t - \int_0^t \Big[\int_0^t (a\tau + b)^2 \exp\Theta_{n-1}(\tau)\, d\tau\Big] d\tau. \tag{12}$$

Some reasonable approximation to the solution of problem (6)-(8) should be taken as the zero approximation. In particular if constant temperatures Θ_{I} and Θ_{II} are specified on the boundaries of the flow, say plane Poiseuille flow, it is expedient to take the linear expression

$$\Theta_0(t) = At + B, \tag{13}$$

as the zero approximation. This is a solution of the problem for constant viscosity ($k = 0$). Then the successive approximations will take account of the heat produced by the dissipation of mechanical energy.

We illustrate the procedure for obtaining successive approximations and determining the constant m_n by a particular example. Suppose the plates limiting plane Poiseuille flow are at the constant temperature $\Theta_0 = 0$. Then

$$\Theta_1(t) = m_1 t - \int_0^t \left[\int_0^t (a\tau + b)^2 d\tau\right] d\tau = m_1 t - \frac{b^2}{2} t^2 - \frac{ab}{3} t^3 - \frac{a^2}{12} t^4. \quad (14)$$

If the first approximation is sufficient we make the solution comply with condition (8) and find for m_1

$$m_1 = \frac{1}{3} k. \quad (15)$$

If the next approximation is required it can be constructed as follows:

$$e^{\Theta_1(t)} \simeq 1 + \Theta_1(t) = 1 + \left(\frac{a^2}{12} + \frac{ab}{3} + \frac{b^2}{2}\right) t - \frac{b^2}{2} t^2 - \frac{ab}{3} t^3 - \frac{a^2}{12} t^4. \quad (16)$$

The validity of this representation is based on the fact that the first approximation, taking account of the heat of friction, is small; i.e., $\Theta_1(t)$ is only slightly different from the zero approximation. Then

$$\Theta_2(t) = m_2 t - \int_0^t \left[\int_0^t (a\tau + b^2) e^{\Theta_1(\tau)} d\tau\right] d\tau = m_2 t - \sum_{i=0}^{6} \frac{A_i t^{i+2}}{(i+1)(i+2)}, \quad (17)$$

where $A_0 = b^2$; $A_1 = 2ab + m_1 b^2$; $A_2 = a^2 + 2abm_1 - b^4/2$; $A_3 = a^2 m_1 - 4ab^3/3$; $A_4 = -11a^2b^2/12$; $A_5 = -a^3b/2$; $A_6 = -a^4/12$. From the condition $\Theta_2(1) = 0$ we find that

$$m_2 = \sum_{i=0}^{6} \frac{A_i}{(i+1)(i+2)}. \quad (18)$$

Thus it is possible to construct the solution of problem (6)-(8) with any preassigned accuracy.

Using the temperature distribution found, the velocity field for the flow of a fluid of variable viscosity between parallel plates is obtained from

$$\frac{dv_1}{dt} = (C_1 - 8t)\, e^{\Theta_1(t)}. \quad (19)$$

Because of the symmetry of the temperature distribution relative to the center of the channel, and consequently also the symmetry of the fluid flow, we find $C_1 = 4$. Then in Eqs. (14) and (16) $a = -4\sqrt{k}$, $b = 6\sqrt{k}$, and finally

$$v_1(t) = \int_0^t 4(1 - 2\tau) \exp\left[m_1\tau - \frac{1}{2} b^2\tau^2 - \frac{1}{3} ab\tau^3 - \frac{1}{12} \tau^4\right] d\tau. \quad (20)$$

Thus by using the method proposed we can reduce the solution of the system of coupled nonlinear momentum and heat flux equations in plane Poiseuille flow to a single quadrature.

The results of calculating the flow of a clear heavy lubricating oil of domestic manufacture [6] in a plane channel are shown in Figs. 1 and 2. This kind of fluid is chosen because its viscosity is strongly temperature dependent.

A comparison of the solution obtained with the result of the numerical integration of the problem of the development of flow in a plane channel [7] shows satisfactory agreement even in the first approximation.

NOTATION

$\Theta = s(T - T_0)$	is the dimensionless temperature;
$\eta = y/h$	is the dimensionless transverse coordinate;
h	is the half-width of the channel;
$k = \mu_0 V_{max}^2 s/\lambda I$; $\varkappa$	are the parameters of the problem;
$\mu = \bar{\mu}/\mu_0$	is the dimensionless viscosity;
$\bar{\mu} = \mu_0 \exp[-s(T - T_0)]$	
μ_0	is the viscosity of the fluid at $T = T_0$;
$V_{max} = -(1/2\mu_0)(dp/dx)h^2$	is the maximum velocity in plane Poiseuille flow at constant viscosity;
I	is the mechanical equivalent of heat;
λ	is the thermal conductivity of the fluid.

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